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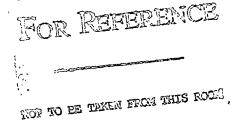
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NUMERICAL EVALUATION OF THE ϵ-INTEGRAL OCCURRING IN

THE THEODORSEN ARBITRARY AIRFOIL POTENTIAL THEORY

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ADVANCE RESTRICTED REPORT

NUMERICAL EVALUATION OF THE ε-INTEGRAL OCCURRING IN

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SUMMARY

A more precise method of evaluating the ε-integral
occurring in the arbitrary airfoil theory of Theodorsen
(NACA Reps. Nos. 411 and 452) is developed by retaining
higher order terms in the Taylor expansion and by use
of Simpson's rule. Formulas are given for routine calcu-
lation of the ε-integral and for the necessary compu-
tational coefficients. The computational coefficients
are tabulated for a 40-point division of the range of
integration from 0 to 2π. With no increase in compu-
tational work the systematic error in the numerical
value of ε is reduced from the order of 1 percent to
approximately 0.1 percent.

INTRODUCTION

The solution of the general problem by means of
conformal transformation for the flow about an arbitrary
airfoil (references 1 and 2), a symmetrical lattice
(reference 3), and a biplane (reference 4) involves the
determination of the imaginary part of a complex trans-
formation function, given the real part. As shown in
references 1 and 2 the real part may be expanded in a
Fourier series and the imaginary part is the conjugate
Fourier series. It is also shown in these references
that the imaginary part ε may be obtained from the
real part ψ by the following functional equation:

$$\epsilon(\phi') = -\frac{1}{2\pi} \int_0^{2\pi} \psi(\phi) \cot \frac{\phi - \phi'}{2} d\phi \qquad (1)$$

This integral occurs frequently in conformal-transformation problems involving the evaluation of the functions on the circle. A procedure for the numerical evaluation of this integral is given in references 1 and 2. This method, which is currently in use at LMAL, gives an error of about 1.5 percent for a 40-point division of the range of integration. An improvement in the accuracy is therefore very desirable, particularly if the labor involved is not increased. A revised method given herein is found to involve a little less work than that previously required and to give an error of only about 0.1 percent. Constants for use in this more precise method have been computed and are presented in table I.

EVALUATION OF THE ϵ-INTEGRAL

The evaluation of the ϵ-integral is complicated by the discontinuity at $\varphi = \varphi'$. This difficulty may be surmounted by a separate solution across the discontinuity. When $s = \varphi - \varphi'$ is substituted in equation (1)

$$\epsilon(\varphi') = -\frac{1}{2\pi} \int_{-\varphi'}^{2\pi-\varphi'} \psi(\varphi' + s) \cot \frac{s}{2} \, ds$$

or, because of the periodicity of this function,

$$\epsilon(\varphi') = -\frac{1}{2\pi} \int_{-s}^{2\pi-s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds \qquad (2)$$

The discontinuity now occurs at $s = 0$. For purposes of numerical evaluation this integral may be broken up as follows:

$$\epsilon(\varphi') = -\frac{1}{2\pi}\left[\int_{-s}^{s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds + \int_{s}^{2\pi-s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds \right]$$

$$= \epsilon_1 + \epsilon_2 \qquad (3)$$

where

$$\epsilon_1 \equiv -\frac{1}{2\pi} \int_{-s}^{s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds \qquad (4)$$

and

$$\epsilon_2 \equiv -\frac{1}{2\pi} \int_{s}^{2\pi-s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds \qquad (5)$$

Evaluation of ϵ_1.- The first integral ϵ_1 includes the discontinuity and the limit s may be taken as some convenient small value. By a Taylor series expansion the integral is easily evaluated as follows:

$$\psi(\varphi' + s) = \psi(\varphi') + s\psi'(\varphi') + \frac{s^2}{2}\psi''(\varphi') + \frac{s}{3!}\psi'''(\varphi') + \dots$$

When this expansion is substituted in equation (4) the integrals containing the even-ordered derivatives are found to be identically zero. Equation (4) then becomes

$$\epsilon_1 = -\frac{1}{2\pi}\left[\psi' \int_{-s}^{s} s \cot \frac{s}{2} \, ds + \frac{\psi'''}{3!} \int_{-s}^{s} s^3 \cot \frac{s}{2} \, ds + \dots\right] \qquad (6)$$

where the derivatives are evaluated at φ'. The Taylor expansion for $\cot \frac{s}{2}$ is

$$\cot \frac{s}{2} = \frac{2}{s} - \frac{s}{6} - \frac{s^3}{360} - \dots$$

and equation (6) is thus obtained as

$$\epsilon_1 = -\frac{1}{2\pi}\left[4s\psi'\left(1 - \frac{s^2}{36} - \frac{s^4}{5600} - \ldots\right) + \frac{2}{9}s^3\psi'''\left(1 - \frac{s^2}{20} - \frac{s^4}{1680} - \ldots\right) \right.$$

$$\left. + \frac{s^5}{150}\psi^V\left(1 - \frac{5s^2}{64} - \ldots\right) + \frac{s^7}{8820}\psi^{VII}\left(1 - \frac{7s^2}{108} - \ldots\right) + \ldots \right] \quad (7)$$

Evaluation of ϵ_2.- The second integral ϵ_2 (equation (5)) may be rearranged for convenience in numerical calculation as follows:

$$\epsilon_2 = -\frac{1}{2\pi}\left[\int_s^\pi \psi(\varphi' + s)\cot\frac{s}{2}\,ds + \int_\pi^{2\pi-s}\psi(\varphi' + s)\cot\frac{s}{2}\,ds \right]$$

$$= -\frac{1}{2\pi}\left[\int_s^\pi \psi(\varphi' + s)\cot\frac{s}{2}\,ds + \int_\pi^s \psi(\varphi' - s)\cot\frac{s}{2}\,ds \right]$$

$$= -\frac{1}{2\pi}\left\{ \int_s^\pi \left[\psi(\varphi' + s) - \psi(\varphi' - s)\right]\cot\frac{s}{2}\,ds \right\} \quad (8)$$

where $-s$ has been substituted for s in the second integral and the limits have been rearranged accordingly.

NUMERICAL METHODS

Method of reference 1.- In reference 1 the interval 0 to 2π is divided into n equal parts of magnitude $2\pi/n$ (n is an even number). The values of ψ are designated $\psi_{-\frac{n}{2}}, \psi_{-\frac{n}{2}+1}, \ldots, \psi_{-1}, \psi_0, \psi_1, \ldots, \psi_{\frac{n}{2}}$

where ψ_0 is the value of ψ at $\varphi = \varphi'$ and $\psi_{-\frac{n}{2}} = \psi_{\frac{n}{2}}$

is the value at $\varphi = \varphi' \pm \pi$. The integrations are performed over intervals of width $2\pi/n$ with the ψ values at the midpoint of the interval. The range of integration for ϵ_1 is from $s = -\pi/n$ to $s = \pi/n$ and for ϵ_2 from π/n to $2\pi - \frac{\pi}{n}$.

The first integral ϵ_1 is evaluated by retaining only the first-order terms in s,

$$\epsilon_1 \cong -\frac{1}{2\pi} 4 s \psi_0' = -\frac{2}{n} \psi_0' \tag{9}$$

where the slope ψ_0' is determined graphically at ψ_0, that is, at $\varphi = \varphi'$ ($s = 0$).

The second integral ϵ_2 is composed of the sum of integrals across each interval

$$\epsilon_2 = -\frac{1}{2\pi} \sum_{k=1}^{n-1} \int_{\frac{2k-1}{n}\pi}^{\frac{2k+1}{n}\pi} \psi(\varphi' + s) \cot \frac{s}{2} \, ds$$

The function ψ does not change much across the interval and is therefore approximated by use of the value at the midpoint $\psi_k \equiv \psi\left(\varphi' + \frac{2k\pi}{n}\right)$. Then,

$$\epsilon_2 \cong -\frac{1}{2\pi} \sum_{k=1}^{n-1} \psi_k \int_{\frac{2k-1}{n}\pi}^{\frac{2k+1}{n}\pi} \cot \frac{s}{2} \, ds$$

$$\cong -\frac{1}{\pi} \sum_{k=1}^{n-1} \psi_k \log \frac{\sin \frac{2k+1}{2n}\pi}{\sin \frac{2k-1}{2n}\pi}$$

or, by equation (8), with $\psi_{n-k} = \psi_{-k}$

$$\epsilon_2 \cong -\frac{1}{\pi} \sum_{k=1}^{n/2} a_k(\psi_k - \psi_{-k}) \qquad (10)$$

where

$$\left.\begin{array}{l} a_k \equiv \log \dfrac{\sin \dfrac{2k+1}{2n}\pi}{\sin \dfrac{2k-1}{2n}\pi} \\[3em] a_{\frac{n}{2}} \equiv 0 \end{array}\right\} \qquad (11)$$

The complete integral is given by $\epsilon = \epsilon_1 + \epsilon_2$, or

$$\epsilon \cong -\left\{\frac{2}{n}\psi_0' + \frac{1}{\pi}\sum_{k=1}^{n/2} a_k(\psi_k - \psi_{-k})\right\} \qquad (12)$$

Values of the constants a_k were given in reference 1 for $n = 10$ and in reference 2 for $n = 20$. Revised values for these constants, together with those for $n = 40$, are given in table II.

Improved method.- The numerical accuracy of the evaluation of the ϵ-integral will be shown to be improved by the following method: The interval 0 to 2π is divided into n equal parts and the ψ values are designated as in the previous section. The second integral ϵ_2 is evaluated by Simpson's rule from ψ_1 to ψ_{n-1} $(\psi_{n-1} = \psi_{-1})$. The range of integration for ϵ_1 is therefore twice as large as that in the previous section, that is, $s = -2\pi/n$ to $2\pi/n$. The approximation in which only the first-order term of equation (7) is used is insufficient and the higher derivatives must be used. These derivatives are most conveniently obtained by numerical differentiation.

The Newton-Stirling formulas for derivatives (reference 5, p. 75) are

$$s\psi' = \delta\psi - \frac{1}{6}\delta^3\psi + \frac{1}{30}\delta^5\psi - \dots$$

$$s^3\psi''' = \delta^3\psi - \frac{1}{4}\delta^5\psi + \dots$$

$$s^5\psi^V = \delta^5\psi - \dots$$

$$\tag{13}$$

where s = tabular interval ($2\pi/n$). The mean central differences $\delta\psi$ can be expressed in terms of the tabular values as

$$2\delta\psi = \psi_1 - \psi_{-1}$$

$$2\delta^3\psi = \psi_2 - 2\psi_1 + 2\psi_{-1} - \psi_{-2}$$

$$2\delta^5\psi = \psi_3 - 4\psi_2 + 5\psi_1 - 5\psi_{-1} + 4\psi_{-2} - \psi_{-3}$$

$$\tag{14}$$

The substitution of relations (13) in equation (7) gives

$$\epsilon_1 = -\frac{1}{2\pi}\left\{2\delta\psi\left(2 - \frac{s^2}{18} - \frac{s^4}{1800} - \dots\right) - 2\delta^3\psi\left(\frac{2}{9} - \frac{s^2}{270} - \dots\right)\right.$$

$$\left. + 2\delta^5\psi\left(\frac{13}{450} - \frac{s^2}{1512} - \dots\right) - \dots\right\}$$

$$\tag{15}$$

The further substitution of relations (14) in equation (15) gives

$$\epsilon_1 = -\frac{1}{2\pi}\left\{(\psi_1 - \psi_{-1})\left[\left(2 + \frac{4}{9} + \frac{19}{90} + \ldots\right)\right.\right.$$

$$\left.- \frac{s^2}{18}\left(1 + \frac{2}{15} + \frac{5}{84} + \ldots\right) - \ldots\right]$$

$$-(\psi_2 - \psi_{-2})\left[\left(\frac{2}{9} + \frac{38}{225} + \ldots\right) - s^2\left(\frac{1}{270} + \frac{1}{378} + \ldots\right) - \ldots\right]$$

$$\left.+(\psi_3 - \psi_{-3})\left[\left(\frac{19}{450} + \ldots\right) - s^2\left(\frac{1}{1512} + \ldots\right) - \ldots\right] - \ldots\right\}$$

or

$$\epsilon_1 \equiv b_1(\psi_{-1} - \psi_1) + b_2(\psi_{-2} - \psi_2) + b_3(\psi_{-3} + \psi_3) \quad (16)$$

where

$$b_1 \equiv \frac{1}{2\pi}\left[\left(2 + \frac{4}{9} + \frac{19}{90}\right) - s^2\left(\frac{1}{18} + \frac{1}{135} + \frac{5}{1512}\right) - \ldots\right]$$

$$= \frac{1}{2\pi}\left(\frac{239}{90} - \frac{167}{2520}s^2 - \ldots\right)$$

$$b_2 \equiv -\frac{1}{2\pi}\left[\left(\frac{2}{9} + \frac{38}{225}\right) - s^2\left(\frac{1}{270} + \frac{1}{378}\right) - \ldots\right] \quad \left.\right\} \quad (17)$$

$$= -\frac{1}{2\pi}\left(\frac{88}{225} - \frac{2}{315}s^2 - \ldots\right)$$

$$b_3 \equiv \frac{1}{2\pi}\left(\frac{19}{450} - \frac{s^2}{1512} - \ldots\right)$$

The second integral ϵ_2 is evaluated by Simpson's rule from ψ_1 to ψ_{n-1}

$$\epsilon_2 \cong -\frac{1}{2\pi} \frac{2\pi}{3n}\left[\psi_1 \cot \frac{\pi}{n} + 4\psi_2 \cot \frac{2\pi}{n}\right.$$

$$\left. + 2\psi_3 \cot \frac{3\pi}{n} + \ldots + \psi_{n-1} \cot \frac{n-1}{n}\pi\right]$$

or, by equation (8), with $\psi_{n-k} = \psi_{-k}$

$$\epsilon_2 \cong \sum_{k=1}^{n/2} c_k(\psi_{-k} - \psi_k) \qquad (13)$$

where

$$\left.\begin{array}{c} c_1 \equiv \dfrac{1}{3n} \cot \dfrac{\pi}{n} \\ \vdots \\ c_k \equiv \dfrac{a}{3n} \cot \dfrac{k\pi}{n} \\ \vdots \\ c_{\frac{n}{2}} \equiv 0 \end{array}\right\} \qquad (19)$$

where (except in the first term) $a = 2$ for k odd and $a = 4$ for k even. The complete integral is given by $\epsilon = \epsilon_1 + \epsilon_2$, or

$$\epsilon \cong \sum_{k=1}^{n/2} A_k(\psi_{-k} - \psi_k) \qquad (20)$$

where

$$A_k = b_k + c_k \qquad (21)$$

Values of A_k for $n = 40$ are given in table I.

ACCURACY OF EVALUATION

The accuracy of the two methods of evaluation described may be determined by integrating various harmonics. The results are presented as ratios of the integrated value to the correct value so that a value of unity is a correct evaluation. Values of this ratio for the harmonics are:

Harmonic	40-point method of reference 1	40-point present method
1	1.01434	0.99977
2	1.02662	1.00044
3	1.03700	.99940
4	1.04547	1.00060
5	1.05280	.99340
10	1.06112	.95420

Inasmuch as the higher harmonics generally enter in a much smaller proportion than the lower ones for such contours as are encountered for airfoil shapes, the error of the 40-point method of reference 1 is of the order of 1.5 percent, whereas that of the new method presented herein is approximately 0.1 percent.

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TABLE I.- VALUES OF A_k FOR USE WITH EQUATION (20)

k	A_k n = 40	k	A_k n = 40
1	0.52827	11	0.01423
2	.14834	12	.02422
3	.07614	13	.01021
4	.10259	14	.01698
5	.04024	15	.00690
6	.06542	16	.01083
7	.02720	17	.00400
8	.04588	18	.00528
9	.01951	19	.00066
10	.03333	20	0

TABLE II.- VALUES OF a_k FOR USE WITH EQUATION (12);

METHOD OF REFERENCES 1 AND 2

k	a_k		
	n = 10	n = 20	n = 40
1	1.06544	1.09057	1.09656
2	.44311	.49426	.50671
3	.23117	.31141	.33028
4	.10302	.21750	.24303
5	0	.15773	.19028
6		.11449	.15453
7		.08022	.12941
8		.05115	.10826
9		.02493	.09207
10		0	.07862
11			.06714
12			.05711
13			.04816
14			.04004
15			.03255
16			.02553
17			.01937
18			.01245
19			.00619
20			0